One-dimensional spherical model with a phase transition

Dorje C. Brody*
The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
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The one-dimensional spherical model is generalized to include long-range interactions. A phase transition is shown to occur for a certain type of interaction. The partition function is explicitly calculated, and the critical behavior of the model is examined.

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I. INTRODUCTION

In order to obtain a phase transition in one dimension, we shall generalize the conventional spherical model which involves only nearest neighbor interactions, to include long-range interactions. First, following Baxter [1], let us briefly review the spherical model solved by Berlin and Kac [2] in 1952. In this model, one assigns a real spin variable σ_j to each lattice site, subject to the constraint $\sum_{j=1}^{N}\sigma_j^2=N$. Thus, the partition function for the spherical model can be written as

$$Z_N=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d\sigma_1\cdots d\sigma_N \times\exp\left\{K\sum_{(j,l)}\sigma_j\sigma_l+H\sum_j\sigma_j\right\} \times\delta\left[N-\sum_j\sigma_j^2\right], \tag{1}$$

where $K=J/kT$, $H=h/kT$, and the summation (j,l) ranges over nearest neighbors only. However, in our modification, the interaction term is replaced by $K\sum_{j,l}\sigma_j J_{jl}\sigma_l$, where J_{jl} is the generic matrix element of $\underline{J}$ in Eq. (3) below, and depends only upon the distance between spins σ_j and σ_l. Thus, the partition function of our model becomes

$$Z_N=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d\sigma_1\cdots d\sigma_N \times\left\{K\sum_{(j,l)}\sigma_j J_{jl}\sigma_l+H\sum_j\sigma_j\right\} \times\delta\left[N-\sum_j\sigma_j^2\right]. \tag{2}$$

With a suitable choice of the matrix $\underline{J}$, this formula can express the partition function of a spherical model on an arbitrary lattice in any dimension. However, since we are now considering the one-dimensional case with periodic boundary conditions, and, as mentioned above, J_{jl} depends on the distance between σ_j and σ_l, the matrix $\underline{J}$ is cyclic, as follows:

$$\underline{J}=\begin{bmatrix} J(0) & J(1) & \cdots & J(N-1)\\ J(N-1) & J(0) & \cdots & J(N-2)\\ \vdots & & \ddots & \vdots\\ J(1) & J(2) & \cdots & J(0)\end{bmatrix}. \tag{3}$$

To ensure translational invariance, i.e., for a finite circular model, rotational invariance, we assume $J(N-n)=J(n)$. Hence, the above matrix is also symmetric. We shall evaluate the thermodynamic functions of systems defined by choosing certain specific expressions for J_{jl}.

II. EVALUATION OF THE PARTITION FUNCTION

We will first follow the derivation by Berlin and Kac [2] as presented by Baxter [1] and then modify this procedure as appropriate for our examples. The partition function of Eq. (2) can be rewritten as

$$Z_N=(2\pi)^{-1}\int_{-\infty}^{\infty}d\sigma_1\cdots d\sigma_N\int_{-\infty}^{\infty}ds\,\exp\left\{K\sum_{(j,l)}\sigma_j J_{jl}\sigma_l+H\sum_j\sigma_j+(a+is)N-(a+is)\sum_j\sigma_j^2\right\}, \tag{4}$$

where we have used the Fourier transform of the δ function and also added an extra term $0=aN-a\sum_j\sigma_j^2$ in the exponent, where a is an arbitrary real constant. Now, we define the matrix $\underline{V}$ by

*Electronic address: d.brody@tp.ph.ic.ac.uk

$$\underline{V}=(a+is)\underline{I}-K\underline{J} \ , \tag{5}$$

where $\underline{I}$ is the $N\times N$ unit matrix. Moreover, defining $\mathbf{H}$ to be an N-dimensional vector with all components equal to H, and $\boldsymbol{\sigma}$ as an N-dimensional vector with elements σ_j, one can write the partition function as

$$Z_N=(2\pi)^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d\boldsymbol{\sigma}\int_{-\infty}^{\infty}ds\,\exp\{-\boldsymbol{\sigma}^T\underline{V}\boldsymbol{\sigma}+\mathbf{H}^T\boldsymbol{\sigma}+(a+is)N\} \ . \tag{6}$$

After diagonalizing the matrix $\underline{V}$ and integrating over $\boldsymbol{\sigma}$, Z_N becomes

$$Z_n=\tfrac{1}{2}\pi^{N/2-1}\int_{-\infty}^{\infty}ds[\det\underline{V}]^{-1/2}\times\exp\{(a+is)N+\tfrac{1}{4}\mathbf{H}^T\underline{V}^{-1}\mathbf{H}\} \ . \tag{7}$$

From the definition of the matrix $\underline{V}$ in Eq. (5), since $\underline{J}$ is a symmetric real cyclic matrix, the eigenvalues of $\underline{V}$ are [3]

$$\lambda_k=a+is-K\left[\sum_{t=1}^{N}J(t)\cos(2\pi kt/N)\right] \ . \tag{8}$$

Thus,

$$\ln\det\underline{V}=\sum_{k=1}^{N}\ln\lambda_k=\sum_{k=1}^{N}\ln\left\{(a+is)-K\left[\sum_{t=1}^{N}J(t)\cos(2\pi kt/N)\right]\right\} , \tag{9}$$

and defining $\omega_k=2\pi k/N$, this becomes

$$\sum_k\ln\{(a+is)-K[J(1)\cos\omega_k+J(2)\cos2\omega_k+\cdots+J(N)\cos N\omega_k]\} \ .$$

The above summation over ω_k can replaced by integration when N is large, and we shall write

$$\ln\det\underline{V}=N[\ln K+g(z)] \ , \tag{10}$$

where

$$g(z)=(2\pi)^{-1}\int_0^{2\pi}d\omega\ln\{z+\xi-[J(1)\cos\omega+J(2)\cos2\omega+\cdots]\} \ , \tag{11}$$

$$z=(a+is-K\xi)/K \ , \tag{12}$$

and $\xi=\sum_{l=1}^{N}J(l)$.

Since $\underline{J}$ is cyclic, $1=(1,\ldots,1)$ is an eigenvector corresponding to the eigenvalue Kz. Hence, $\mathbf{H}$ is the eigenvector of $\underline{V}$ corresponding to the eigenvalue $a+is-K\xi=Kz$ and the second term in the exponent of Eq. (7) becomes $NH^2/4Kz$. Substituting this and the above equations (11) and (12) into (7), we can now write the partition function as

$$Z_N=(K/2\pi i)(\pi/K)^{N/2}\int_{c-i\infty}^{c+i\infty}dz\,\underline{e}^{Nf(z)} \ , \tag{13}$$

where

$$f(z)=Kz+K\xi-\tfrac{1}{2}g(z)+\frac{H^2}{4Kz} \tag{14}$$

and $c=(a-K\xi)/K$. Evaluating the above integral by the method of steepest descent, the partition function for sufficiently large N becomes

$$Z_N=\frac{K}{2\pi}\left[\frac{\pi}{K}\right]^{N/2}\underline{e}^{Nf(z_0)} \ , \tag{15}$$

where z_0 is the value of z at which $f'(z)=0$ and the arbitrary constant a has been chosen to be $K(z_0+\xi)$ (i.e., $c=z_0$). From the above equation (14), the condition $f'(z_0)=0$ is equivalent to

$$K-\frac{H^2}{4Kz_0^2}=\tfrac{1}{2}g'(z_0) \ . \tag{16}$$

III. EVALUATION OF THERMODYNAMIC FUNCTIONS

From the expression (15) for the partition function, the free energy per spin ψ in the thermodynamic limit becomes

$$-\frac{\psi}{kT}=\lim_{N\to\infty}N^{-1}\ln Z_N=\tfrac{1}{2}\ln\left[\frac{\pi}{K}\right]+f(z_0) \ . \tag{17}$$

Hence, the magnetization, magnetic susceptibility, internal energy, and specific heat per spin are

$$M=-\frac{d}{dH}\left[\frac{\psi}{kT}\right]=\frac{H}{2Kz_0}=\frac{h}{2Jz_0} \ , \tag{18}$$

$$\chi=\frac{\partial M}{\partial h}=\frac{1}{2Jz_0}-\frac{h}{2Jz_0^2}\left[\frac{\partial z_0}{\partial h}\right] , \tag{19}$$

$$U=-T^2\frac{\partial}{\partial T}\left[\frac{\psi}{T}\right]=\tfrac{1}{2}kT-J(z_0+\xi)-\frac{h^2}{4Jz_0} \ , \tag{20}$$

and

$$C=\frac{\partial U}{\partial T}=\tfrac{1}{2}k-J\left[\frac{\partial z_0}{\partial T}\right]+\frac{h^2}{4Jz_0^2}\left[\frac{\partial z_0}{\partial T}\right] . \tag{21}$$

z_0 is a function of K and H (in particular, a function of T) and is determined from Eqs. (11) and (16). We shall examine a special case and evaluate z_0 as a function of K and H in Sec. IV.

IV. A SPECIAL CHOICE OF J

In the previous sections we have described the derivation of the partition function and some thermodynamic functions for an arbitrary translation-invariant one-dimensional spherical model. However, we have not specified the generic matrix element $J(l)$ of $\underline{J}$, i.e., the interaction of spins σ_j and σ_{j+l}. Thus, we have as many one-dimensional spherical models with distance-

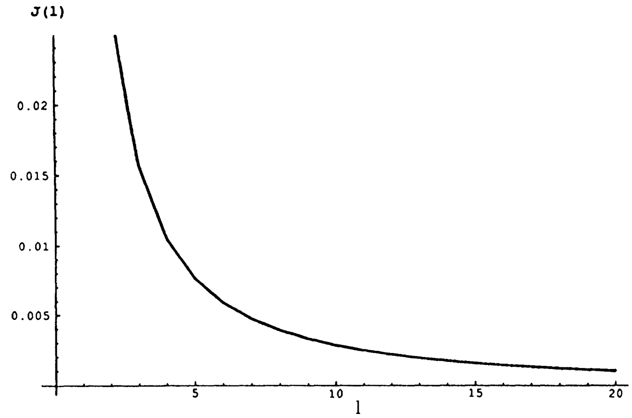

FIG. 1. Interaction strength between spins.

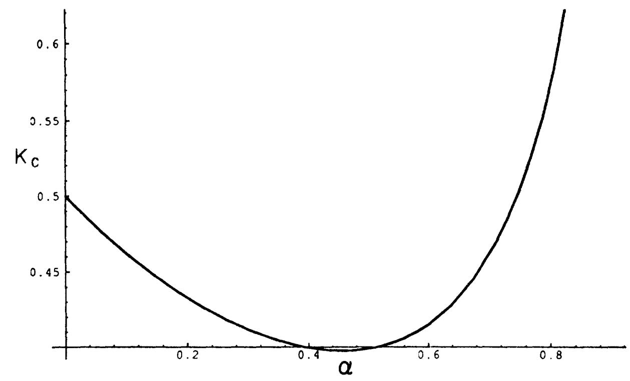

FIG. 2. Critical temperature as a function of α.

dependent couplings as choices of $\underline{J}$. If we choose $J(l)=\exp(-l^2/4)$, the interaction becomes Gaussian, and if we choose $J(1)=1$ and $J(l)=0$ $(l>1)$, we recover the nearest neighbor interaction model, and so on.

However, we must suitably restrict our choice of $\underline{J}$ in order to obtain a phase transition. From Eq. (11), the right-hand side of (16) becomes

$$\tfrac{1}{2}g'(z_0)=\tfrac{1}{2}\left[\frac{\partial g(z)}{\partial z}\right]_{z=z_0}$$
$$=(4\pi)^{-1}\int_0^{2\pi}d\omega\left[z_0+\xi-\sum_{l=1}^{N}J(l)\cos(l\omega)\right]^{-1} . \tag{22}$$

Following Baxter [1] there exists a critical point $K_c=\frac{1}{2}g'(0)$ only if this integral converges. Thus, if we define $j(\omega)$, for large N, to be

$$j(\omega)=\sum_{l=1}^{N}J(l)\cos(l\omega) , \tag{23}$$

then, in order to obtain a finite nonzero critical temperature, $j(\omega)$ must be of the form ω^α as $\omega\downarrow 0$, where $0<\alpha<1$. Although the above integral also converges for $\alpha\le 0$, the value of α must be positive since negative values of α correspond to negative temperatures. Hence, for the simplest choice, each element of the generic matrix $J(l)$ becomes a Fourier coefficient of ω^α, that is,

$$J(l)=-\frac{1}{2\pi}\int_0^{2\pi}\omega^\alpha\cos(l\omega)d\omega , \tag{24}$$

or, by a known formula [4],

$$J(l)=\frac{1}{4\pi l^{1+\alpha}}\{e^{i(1+\alpha)\pi/2}\Gamma(1+\alpha,-2l\pi i)$$
$$+e^{-i(1+\alpha)\pi/2}\Gamma(1+\alpha,2l\pi i)$$
$$-2\cos[(1+\alpha)\pi/2]\Gamma(1+\alpha)\} , \tag{25}$$

where $\Gamma(x,y)$ denotes the incomplete Γ function. From the above equation (24), $J(l)$ is a positive, monotonic decreasing function of the lattice distance l. The behavior of $J(l)$, i.e., the strength of the coupling, for the value $\alpha=\frac{1}{2}$, is shown in Fig. 1. The α dependence of the critical point $K_c=\frac{1}{2}g'(0)$ can be obtained by substituting $j(\omega)=\omega^\alpha$ into Eq. (22) above, which yields

$$K_c=\frac{(2\pi)^{-\alpha}}{2(1-\alpha)} . \tag{26}$$

Hence, the maximum critical temperature $T_{c\max}$ is obtained for $\alpha\cong 0.46$, and the value of $T_{c\max}$ is

$$T_{c\max}=\frac{J(2\pi)^{1/2}}{k} . \tag{27}$$

$T_c\to 0$ for both limiting cases, $\alpha\to 0$ and $\alpha\to 1$, since $J(l)=0$ in the limit $\alpha\to 0$. The behavior of K_c as a function of α is shown in Fig. 2.

The explicit expression for the partition function can be obtained by evaluating the integral in Eq. (11) and substituting the result into Eq. (15), which yields

$$Z_N=\frac{K}{2\pi}\left[\frac{\pi}{K}\right]^{N/2}[z_0+(2\pi)^\alpha]^{-1/2}\exp\left\{N\left[K\left[z_0+\frac{(2\pi)^\alpha}{1+\alpha}\right]+\frac{1}{2}\left[\alpha-{}_2F_1(1,\alpha^{-1};1+\alpha^{-1};-z_0^{-1}(2\pi)^\alpha)+\frac{H^2}{2Kz_0}\right]\right]\right\} , \tag{28}$$

where ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function.

V. BEHAVIOR OF THERMODYNAMIC FUNCTIONS

Now, we shall examine the behavior of the thermodynamic functions in the limit $h\to 0$. First, consider the temperature range $T>T_c$ $(K<K_c)$. Then, it follows from Eqs. (16) and (22) that z_0 approaches a nonzero value w in the limit $h\to 0$ $(H\to 0)$. Also, as in Baxter [1] (p. 67), w is a monotonic function of T for $T\ge T_c$ and approaches zero as $T\to T_c$. Thus, the thermodynamic functions in the limit $h\to 0$ become

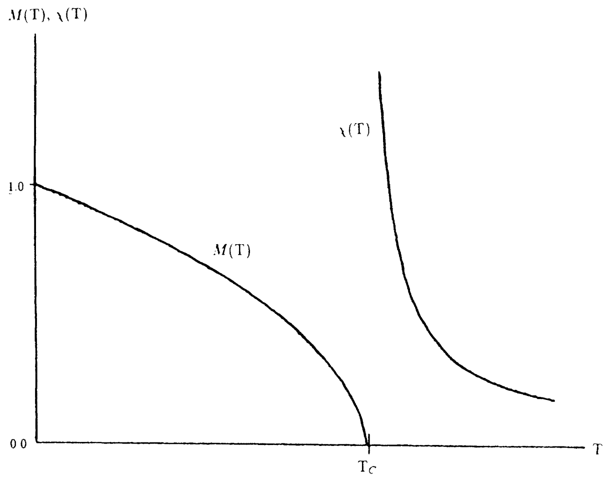

FIG. 3. Spontaneous magnetization M and magnetic susceptibility χ.

$$M=0\ , \tag{29}$$

$$\chi=\frac{1}{2Jw}\ , \tag{30}$$

$$U=\tfrac{1}{2}kT-J(w+\xi)\ , \tag{31}$$

and

$$C=\tfrac{1}{2}k-J\left[\frac{\partial w}{\partial T}\right]\ , \tag{32}$$

where $\xi=(2\pi)^{\alpha}/(1+\alpha)$. No spontaneous magnetization occurs above the critical temperature and since $w\to 0$ as $T\to T_c$, the magnetic susceptibility diverges at the critical point.

For $T<T_c$, $z_0\to 0$ as $h\to 0$. Hence, from Eq. (16),

$$\lim_{H\to 0}\frac{H}{z_0}=[4K(K-K_c)]^{1/2}=2K\left[1-\frac{T}{T_c}\right]^{1/2}\ . \tag{33}$$

Hence,

$$M=\left[1-\frac{T}{T_c}\right]^{1/2}\ , \tag{34}$$

$$\chi\to\infty\ , \tag{35}$$

$$U=\tfrac{1}{2}kT-J\xi\ , \tag{36}$$

and

$$C=\tfrac{1}{2}k\ . \tag{37}$$

The behaviors of the magnetization and magnetic susceptibility are shown in Fig. 3.

VI. OTHER ONE-DIMENSIONAL MODELS WITH PHASE TRANSITIONS

The Ising model with the interaction energy

$$J(l)=Jl^{-s} \tag{38}$$

is known [5] to exhibit a phase transition for $1<s<2$ but not for $s>2$ or $s\le 1$. Thus, $s=2$ is the critical power dividing short- and long-range interactions in the one-dimensional model. In particular, the borderline case $s=2$ (the Anderson model) has been extensively studied in connection with time evolution phenomena such as the Kondo problem.

In particular, Dyson [6] showed that, in the Ising model with the interaction energy

$$J(l)=J\left[\frac{\ln\ln(|l|+3)}{l^2}\right]\ ,$$

the order parameter is discontinuous at the critical point, and a rigorous proof of the existence of a phase transition for the Anderson model ($s=2$) was given by Frohlich and Spencer [7] However, that work, although proving that the critical point $\beta_c<\infty$, did not present any specific upper bound for β_c. The upper bound on β_c as well as the existence of the spontaneous magnetization, assuming a translation-invariant interaction such that the limit

$$\lim_{l\to\infty} l^2J(l)=J^+\ ,$$

is well defined, has been studied in detail, using results from percolation theory, by Aizenman *et al.* [8].

The latter authors also rigorously proved that for the on-dimensional q-state Potts model, there exists a positive spontaneous magnetization $M(\beta)$ if $\beta J^+>q$, where J^+ is defined above, that this magnetization is discontinuous at the transition point, and that $M(\beta)=0$ if $\beta J^+\le 1$. Notice that the behavior of the magnetization M in the present model is different from that of the above-mentioned models in that the magnetization is continuous (although the derivative of the magnetization is not continuous) in the present model but not in the above-mentioned models.

VII. CONCLUSION

The critical behavior of the present model is similar to that of the conventional spherical model in dimensions greater than 2. This follows from the fact that the integral in Eq. (22) converges for dimensions greater than 2 in the conventional spherical model. However, in the present model, the introduction of long-range interactions eliminates the need to consider higher dimensions in order to exhibit critical behavior.

As was mentioned in Sec. VI, there are other one-dimensional spin models which exhibit phase transitions at finite temperature. Dyson's hierarchical model [9] is one such model, but is rather artificial in lacking translational invariance. The one-dimensional Ising model with a translation-invariant coupling of the asymptotic form $J(l)=l^{-2}$ (the Anderson model) and the q-state Potts model with the same interaction also display a discontinuity in the spontaneous magnetization. Although the Anderson model has been extensively studied in connection with the Kondo problem, the mathematical aspects are highly nontrivial. By contrast, the mathematical derivations involved in the present model are quite straightforward. Also, the present interaction is translation invariant and decreases monotonically with increasing distance between spins. Application of general theorems on Fourier series [10] shows that the series $\sum_{l=0}^{\infty} J(l)$ is absolutely convergent, in particular, $J(l)=o(l^{-1})$ as $l \to \infty$. Thus, the coupling introduced in the present case also appears quite natural and physically plausible as well as the above-mentioned Anderson and Potts models.

One may expect that the introduction of suitable long-range interactions in a two-dimensional spherical model would also yield a phase transition, and moreover, that the required coupling strength would decrease even more rapidly than in the preceding one-dimensional case. The two-dimensional case will be discussed in a subsequent paper.

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